Polarimetry Statistics

Summary of arXiv:2310.20196

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Likelihood Formulation (Marshall 2021)

See 12/1/20 CalStats WG presentation

- Expected counts in $dEd\psi$ in time T for $Q = qI, U = uI$:
  - $\lambda(E, \psi; n_0, q, u)dEd\psi = [1 + \mu_E(q\cos 2\psi + u\sin 2\psi)]n_E A_E T dEd\psi$, where
    - A_E is the instrument effective area (independent of q or u by definition)
    - $n_E = n_0\phi(E)$ has units of ph/cm²/s/keV per unit (measured) phase angle, ψ
  - Require $\Pi^2 \equiv q^2 + u^2 \leq 1$ physically (P is fractional linear polarization)
  - Define $\phi_0 = \tan^{-1}(u/q) = 2\varphi$
- Likelihood: $S(n_0, q, u) = -2\ln\mathscr{L}$ or $\tilde{S}(q, u) = -2\sum_i \ln(1 + q\mu_i\cos 2\psi_i + u\mu_i\sin 2\psi_i)$
- MDP99 $= 4.29/\sqrt{\sum \mu_E^2 C(E)}$ for small Pm

Method for fitting with Background

- Start with likelihood method that accounts for unpolarized BG:

  $$-2\ln\mathcal{L} = S = -2\sum_{i=1}^{N}\ln[N_0(1+qc_i+us_i)+\zeta B] + 2N_0 + 2B(1+\zeta) - 2N_B\ln B$$

  where $c_i = \mu_i\cos 2\psi_i$, $s_i = \mu_i\sin 2\psi_i$, $\zeta = \Omega_S/\Omega_B$, N (N_B) is count in src (bg) region, N_0 is the expected source counts in the src region

- Then $\hat{B} = N_B$, $\hat{N}_0 = \sum_{i=1}^{N}[1+\hat{\Pi}\mu_i\cos(2\psi_i+2\hat{\varphi}) + \frac{\zeta N_B}{\hat{N}_0}]^{-1}$, which can be solved for $\hat{N}_0$, given trial values of $\hat{\Pi} = \sqrt{\hat{q}^2+\hat{u}^2}$ and $\tan 2\hat{\varphi} = \hat{u}/\hat{q}$

- Then $\sigma_{\Pi}^2 = \frac{N^2}{N_0^2\sum_i c_i^2} \approx \frac{2N}{C_S^2\langle\mu^2\rangle}$ and $\mathrm{MDP}_{99} \longrightarrow \frac{4.29\sqrt{(C_S+C_B)}}{C_S\sqrt{\langle\mu^2\rangle}}$ as $\hat{\Pi} \longrightarrow 0$

Method for fitting with Polarized Background

- As before, let q_b and u_b be the normalized q and u for the BG:

$$S = -2\sum_{i=1}^{N} \ln[N_0(1 + qc_i + us_i) + \zeta B(1 + q_b c_i + u_b s_i)] + 2N_0 + 2B(1+\zeta) - 2N_B \ln B + -2\sum_{j=1}^{N_B} \ln(1 + q_b c_j + u_b s_j)$$

- Then $\hat{B} = N_B$, $\hat{q}_b \approx \sum_j c_j / \sum_j c_j^2$, $\hat{u}_b \approx \sum_j s_j / \sum_j s_j^2$ from BG region

- $$\sigma_\Pi^2 \longrightarrow \frac{1}{N_0^2}\left[\frac{N^2}{\sum_i c_i^2} + \frac{\zeta^2 N_B^2}{\sum_j c_j^2} + \zeta^2 N_B \Pi_B^2 \cos^2(2\phi - 2\phi_B)\right] \text{ as } \hat{\Pi} \longrightarrow 0$$

- 1st term: same as in unpolarized case
- 2nd term: independent of BG polarization: results from bg pol'n uncertainty
- 3rd term: depends on BG polarization direction; largest along BG EVPA, zero at 45°
- Caveat: requires $\zeta^2 N_B \ll N$ and $\zeta^2 \Pi_B^2 \langle \mu^2 \rangle \ll 1$

Stokes Method with Polarized Background

- Procedure: Subtract independent Stokes of BG region from Source region
- $S_s = (I, qI, uI, 0), S_b = (I_b, q_b I_b, u_b I_b, 0), S_t = S_s - \zeta S_b$
- Want variance of $\Pi_t = \sqrt{(qI - q_b I_b)^2 + (uI - u_b I_b)^2} / (I - I_b)$
- Determine variances in Stokes for regions:
  - $V(S_s) = (N, Nq^2 + \frac{N^2}{\sum_i \mu_i^2 c_i^2}, Nu^2 + \frac{N^2}{\sum_i \mu_i^2 s_i^2}, 0)$
  - Compute $V(S_b)$ for BG region, then $V(S_t) = V(S_s) + \zeta^2 V(S_b)$
  - Let $q = \Pi \cos 2\phi, u = \Pi \sin 2\phi, q_b = \Pi_b \cos 2\phi_b, u_b = \Pi_b \sin 2\phi_b, P_t^2 = Q_t^2 + U_t^2$
- Then $V(\Pi) = \frac{V(P_t)}{I_t^2} + \frac{\Pi^2}{I} = \frac{V(P_t)}{(N - \zeta N_B)^2} = \frac{1}{(N - \zeta N_B)^2} \left[\frac{Q_t^2}{P_t^2} V(Q_t) + \frac{U_t^2}{P_t^2} V(U_t) \right]$ as $\Pi \longrightarrow 0$

Stokes Method with Polarized Background

- Procedure: Subtract independent Stokes of BG region from Source region
- So far: $V(\Pi) = \frac{1}{(N - \zeta N_B)^2} \left[\frac{Q_t^2}{P_t^2} V(Q_t) + \frac{U_t^2}{P_t^2} V(U_t) \right]$
- Then

$$V(\Pi) \approx \frac{1}{(N - \zeta N_B)^2} \left[\frac{N^2}{\sum_i \mu_i^2 c_i^2} + \frac{\zeta^2 N_B^2}{\sum_j \mu_j^2 c_j^2} + \zeta^2 N_B \Pi_B^2 (\cos^2 2\phi \cos^2 2\phi_b + \sin^2 2\phi \sin^2 2\phi_b) \right]$$

- $\sigma_\Pi^2 \longrightarrow \frac{1}{N_0^2} \left[\frac{N^2}{\sum_i \mu_i^2 c_i^2} + \frac{\zeta^2 N_B^2}{\sum_j \mu_j^2 c_j^2} + \zeta^2 N_B \Pi_B^2 \cos^2(2\phi - 2\phi_B) \right]$ as $\hat{\Pi} \longrightarrow 0$
- Caveat: requires $\sum_i \mu_i^2 c_i^2 \approx \sum_i \mu_i^2 s_i^2$ and $\sum_j \mu_j^2 c_j^2 \approx \sum_j \mu_j^2 s_j^2$

Summary

- Variances are underestimated if BG polarization is assumed to be zero
  - Traditional formula is valid only for known unpolarized BG
  - OK if there are external reasons to assume unpolarized BG
  - Generally, should assume it is possibly polarized
- Error contours will depend on polarization EVPA
  - Effect may be small unless Π_b and/or $\zeta = \Omega_s/\Omega_b$ are large
  - Will only increase uncertainty