Statistics in X-ray Polarimetry

Overview of some statistics in use or development
for X-ray Polarimetry

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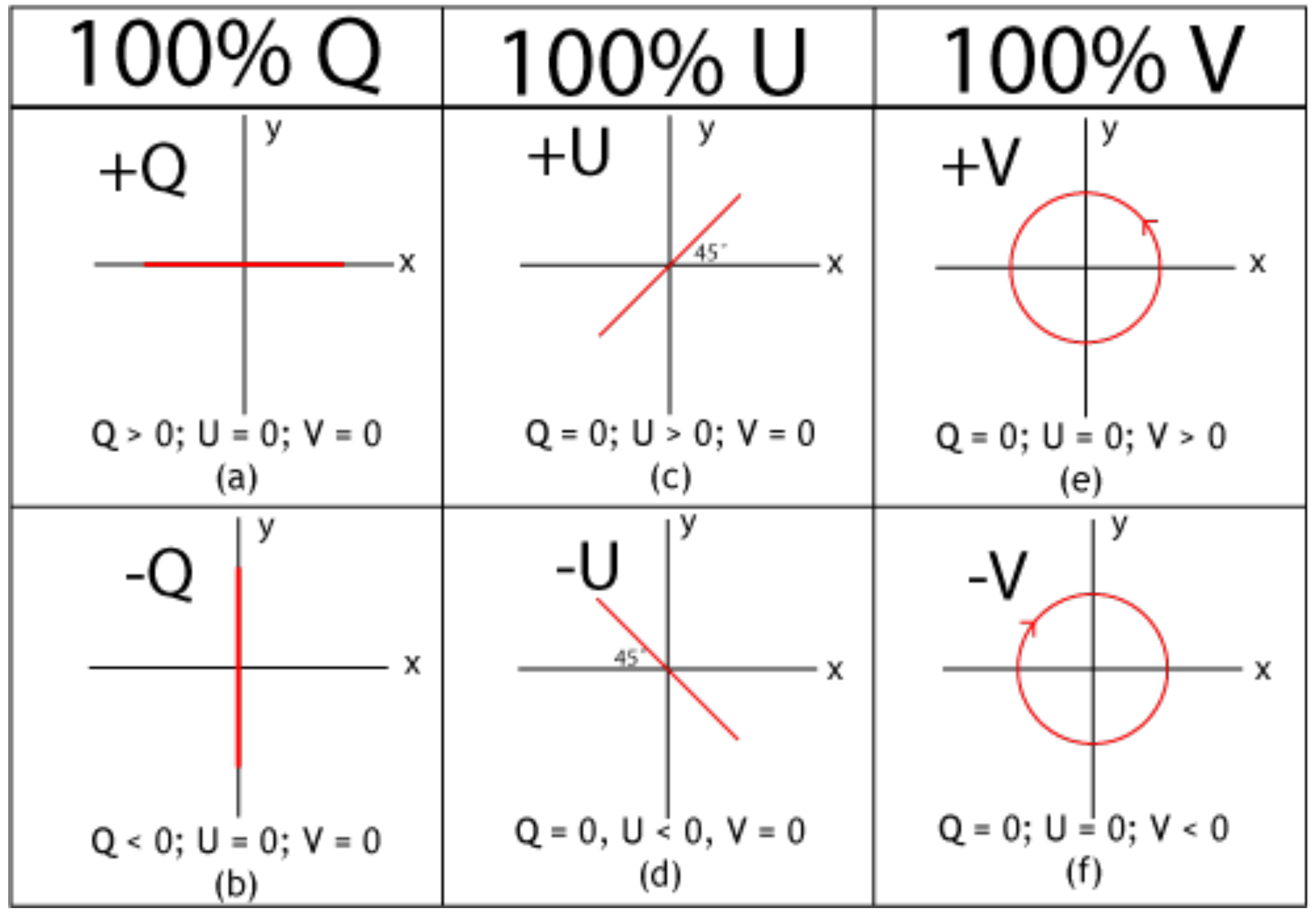
A Few Basics

- Stokes parameters are handy:
  - I = total intensity
  - Q, U are orthogonal linearly polarized parts
  - V is circular (+ or -) polarized intensity
- Common alternative: Π, ϕ
  - $\Pi = (Q^2 + U^2)^{1/2} / I$
  - $\phi = \tan^{-1}(U/Q) = 2 \times \mathrm{EVPA}$

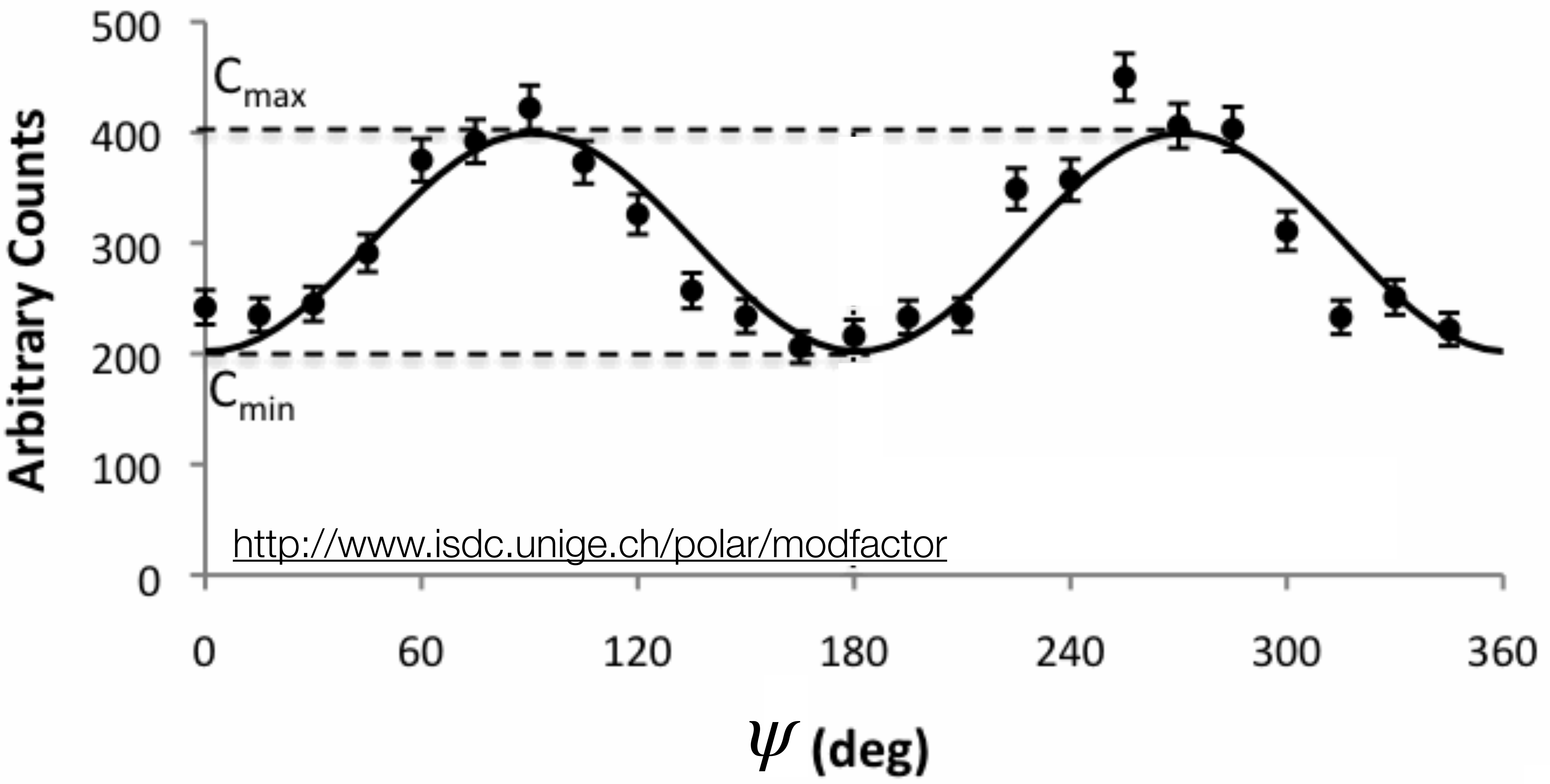


- A beam is "unpolarized" if the photon **set** is randomly polarized (Π = V = 0)
- MDP = 'Minimum Detectable Polarization' (at 99% conf.) = $\dfrac{4.292\sqrt{N_S + N_B}}{\mu N_S}$

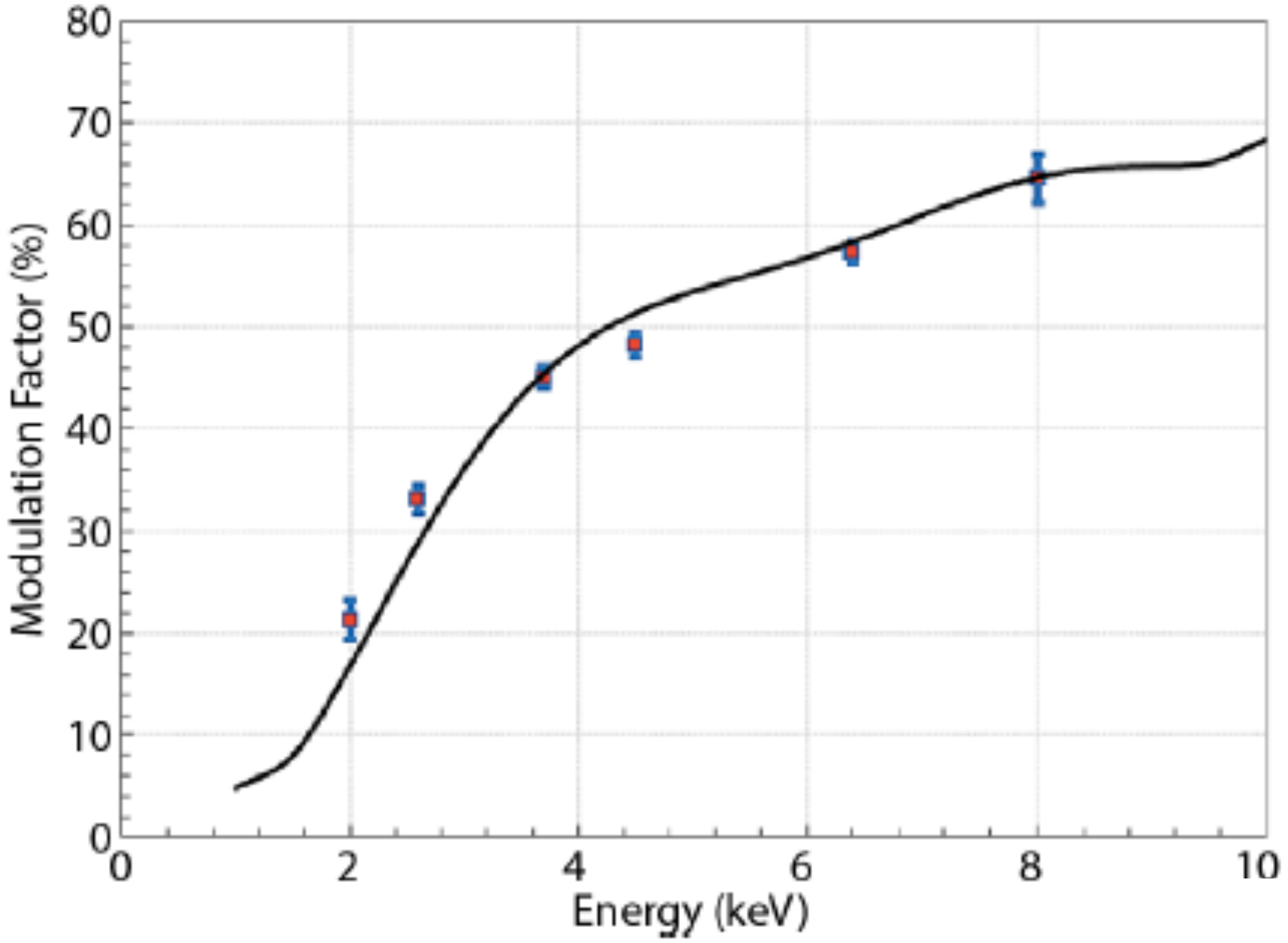
$$4.292 = 2(-\log[0.01])^{1/2}$$

Modulation of Polarized Signals

Modulation Curve 100% polarized source



IXPE Modulation Curve



Modulation Factor = μ = (C_{max}-C_{min})/(C_{max}+C_{min})

$$f(\psi) = \frac{1}{2\pi}\left(1 + p_0\,\mu\cos(2(\psi - \psi_0))\right).$$

Relevant Work

- Elsner, O'Dell, & Weisskopf (2012): Gaussian statistics, BG $\mathrm{MDP}_{99} = \frac{4.292\sqrt{N_S + N_B}}{\mu N_S}$
- Kislat+ (2015): Unbinned analysis, event weighting
- ★ Strohmayer (2017): Fitting IQU spectra in xspec, mRMF $= \mu R(E; E')$
- Burgess+ (2019): Likelihood method for GRB polarimetry
- Peirson+ (2021): Machine learning to get better μ
- Marshall (2021, 2022): Likelihood method, modeling μ

} See 12/1/20 CalStats WG presentation

- ★ Di Marco+ (2022): Event weights using IXPE track ellipticities α
- Gonzales-Caniulef+ (2022): Likelihood method for pulsars
- ★ Marshall (in prep.): Likelihood with BG, nonuniform ψ, mRMF(E, α; E')

How do we fit IXPE data?

- Split spectra into I, Q, U
- E is "observed", E' is "true" and unknown
- Need RMFs, $R(E', E)$. and ARFs, $\epsilon(E')$
- Assumes mRMF = $\mu(E')R(E', E)$
- Complication: $\mu = f(\alpha)$, α = ellipticity (Di Marco+ 2022)

$$I(E) = \int_{E'} F(E')\epsilon(E')R(E', E)dE' \tag{25}$$

$$U(E) = \int_{E'} W(E')\mu(E')\epsilon(E')R(E', E)dE' \tag{26}$$

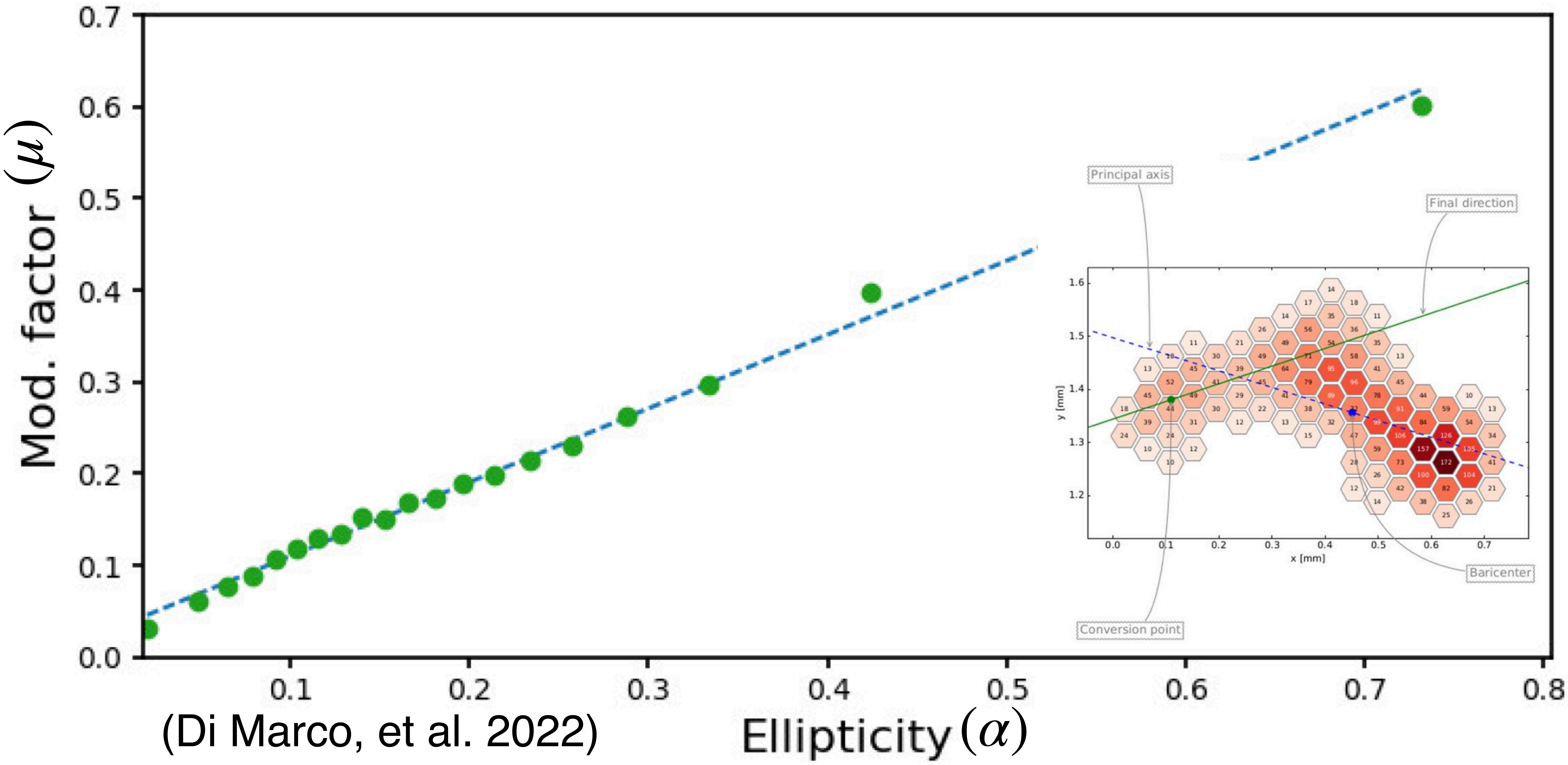
$$Q(E) = \int_{E'} Z(E')\mu(E')\epsilon(E')R(E', E)dE' \; . \tag{27}$$

$$O(E, \psi) = I(E) + U(E)\sin(2\psi) + Q(E)\cos(2\psi) \; . \tag{28}$$

One model spectrum, $F(E')$, is folded through the full detector response function, $\epsilon(E')R(E', E)$, and the two new spectra, $W(E') = F(E')a(E')\sin(2\psi_0'(E'))$ and $Z(E') = F(E')a(E')\cos(2\psi_0'(E'))$ are folded through the "modulated response" function, $\mu(E')\epsilon(E')R(E', E)$.

Strohmayer (2017)

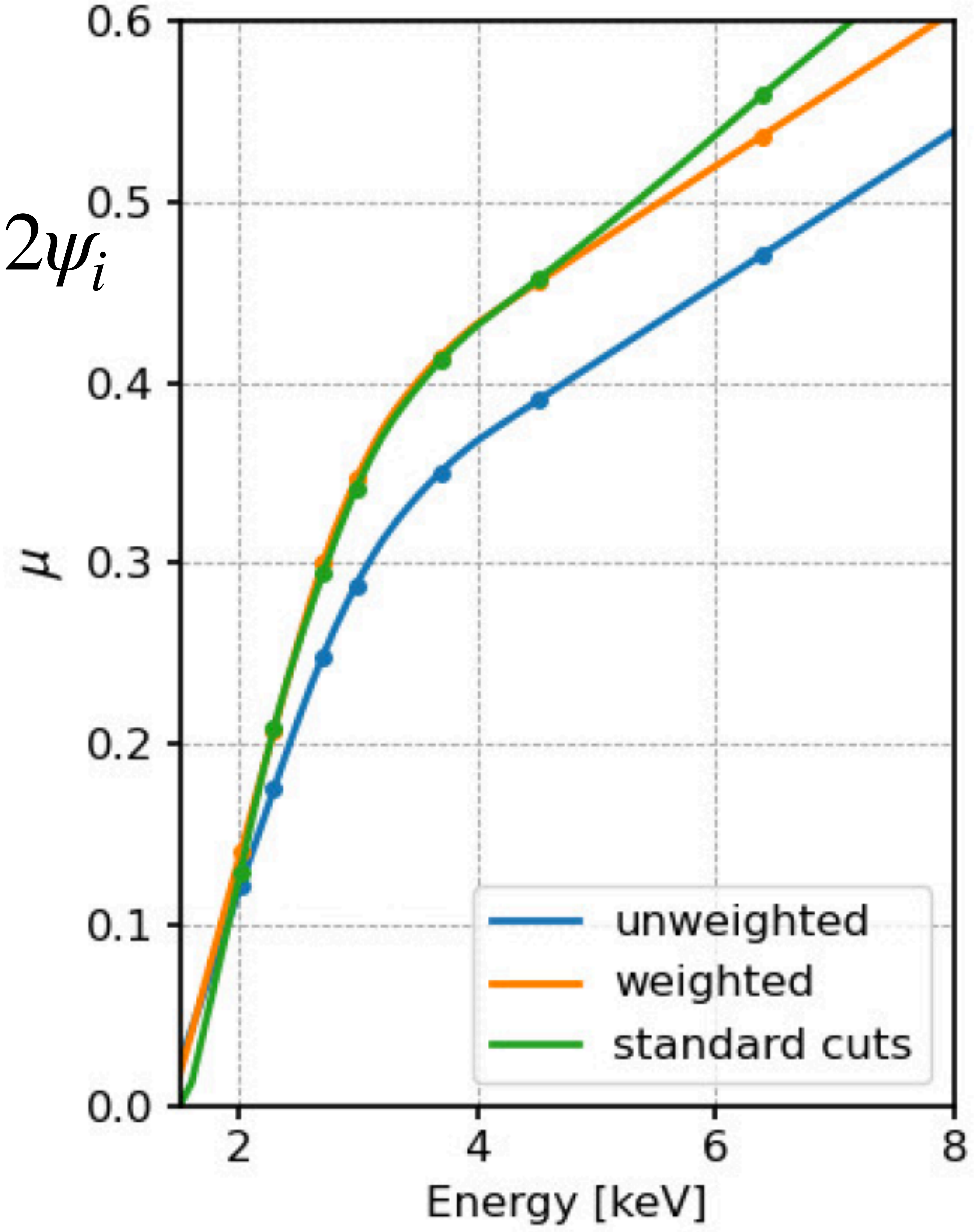
Inferring the Modulation Factor



(Di Marco, et al. 2022)

IXPE Ellipticity Weighting (Di Marco+ 2022)

- Compute $w_i = \alpha_i^{0.75}$ for each event
- Estimate I,Q,U:
  $$\mathcal{I} = \sum w_i, \quad \mathcal{Q} = 2\sum w_i \cos 2\psi_i, \quad \mathcal{U} = 2\sum w_i \sin 2\psi_i$$
- Compute $\hat{\Pi} = \frac{\sqrt{\mathcal{Q}^2 + \mathcal{U}^2}}{\mu \mathcal{I}}$ with uncertainty
  $$\sigma_\Pi^2 \simeq \frac{2\sum w_i^2}{\mu^2 \mathcal{I}^2} = \frac{2}{\mu^2 N_{\rm eff}} \quad (\text{for } \Pi \ll 1)$$
- Develop "weighted" modulation functions
- Weighted MDPs are ~5% better than standard



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- Suggestion: mRMF for J (=3-10) values of α_j
  $\mathscr{M}_j(E', E) = \mu(\alpha_j, E')\epsilon(E')\phi(\alpha_j, E')R(E', E)$

Q: is μ dependent only on α or on both α and E'?

One model spectrum, $F(E')$, is folded through the full detector response function, $\epsilon(E')R(E', E)$, and the two new spectra, $W(E') = F(E')a(E')\sin(2\psi_0'(E'))$ and $Z(E') = F(E')a(E')\cos(2\psi_0'(E'))$ are folded through the "modulated response" function, $\mu(E')\epsilon(E')R(E', E)$.

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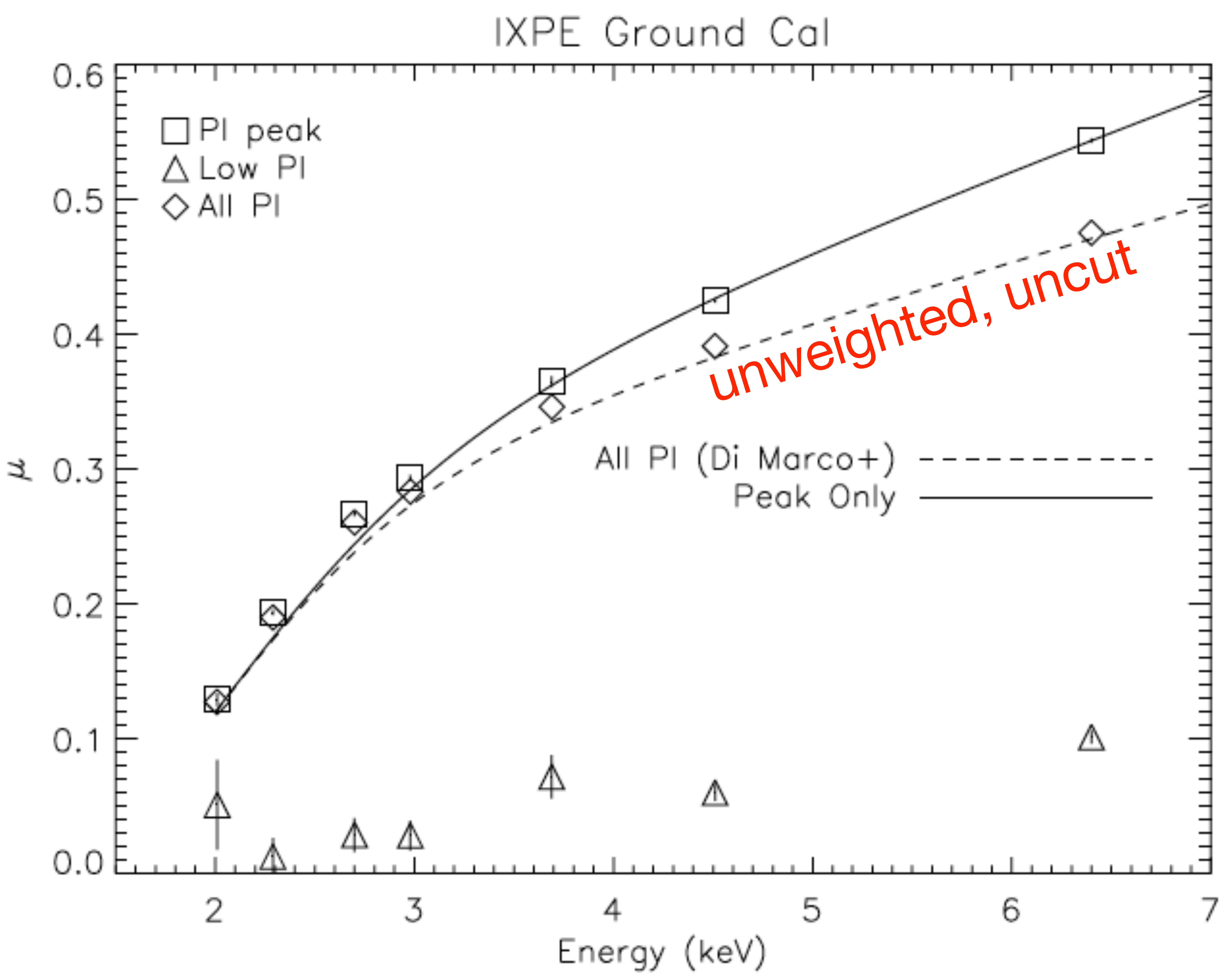
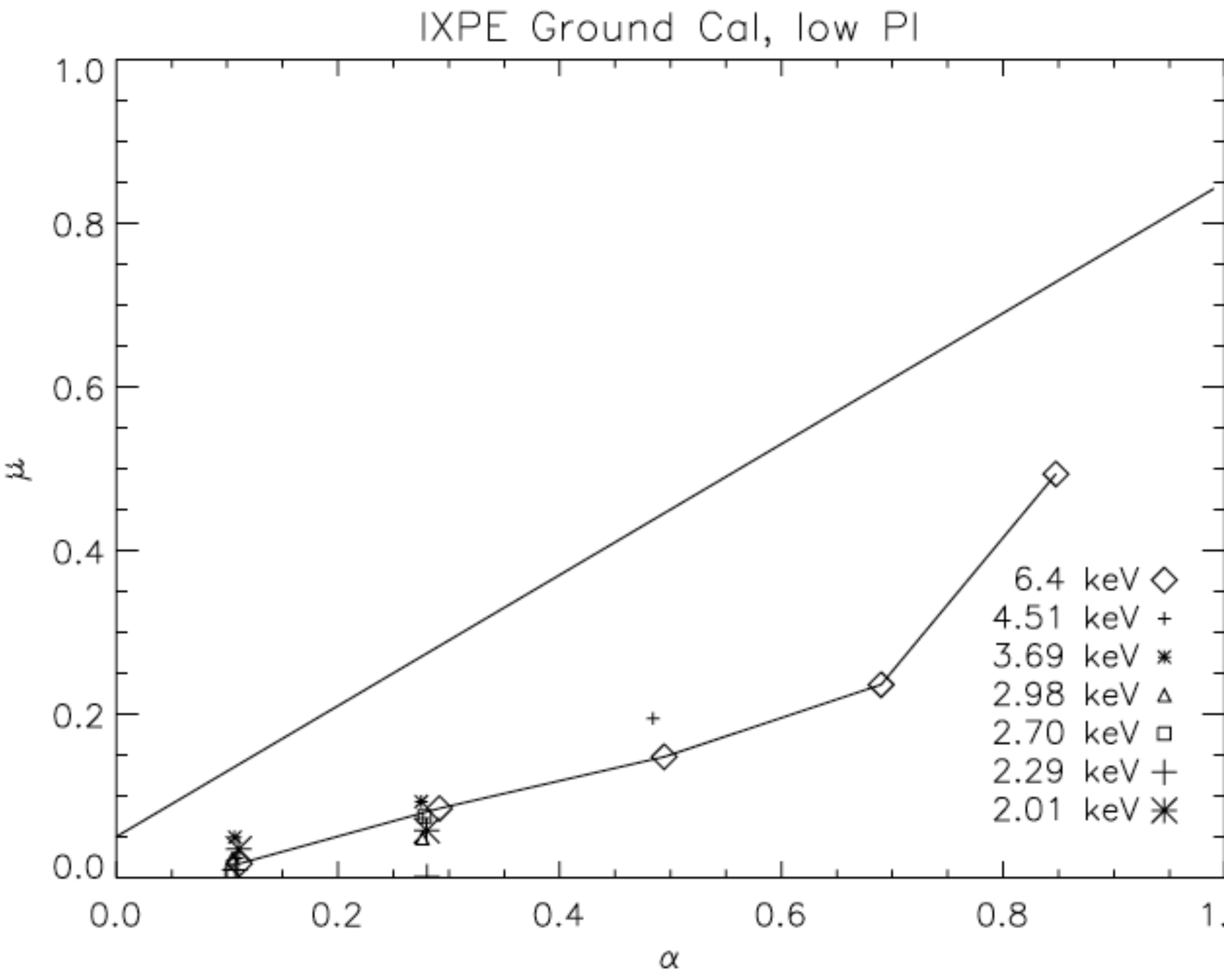
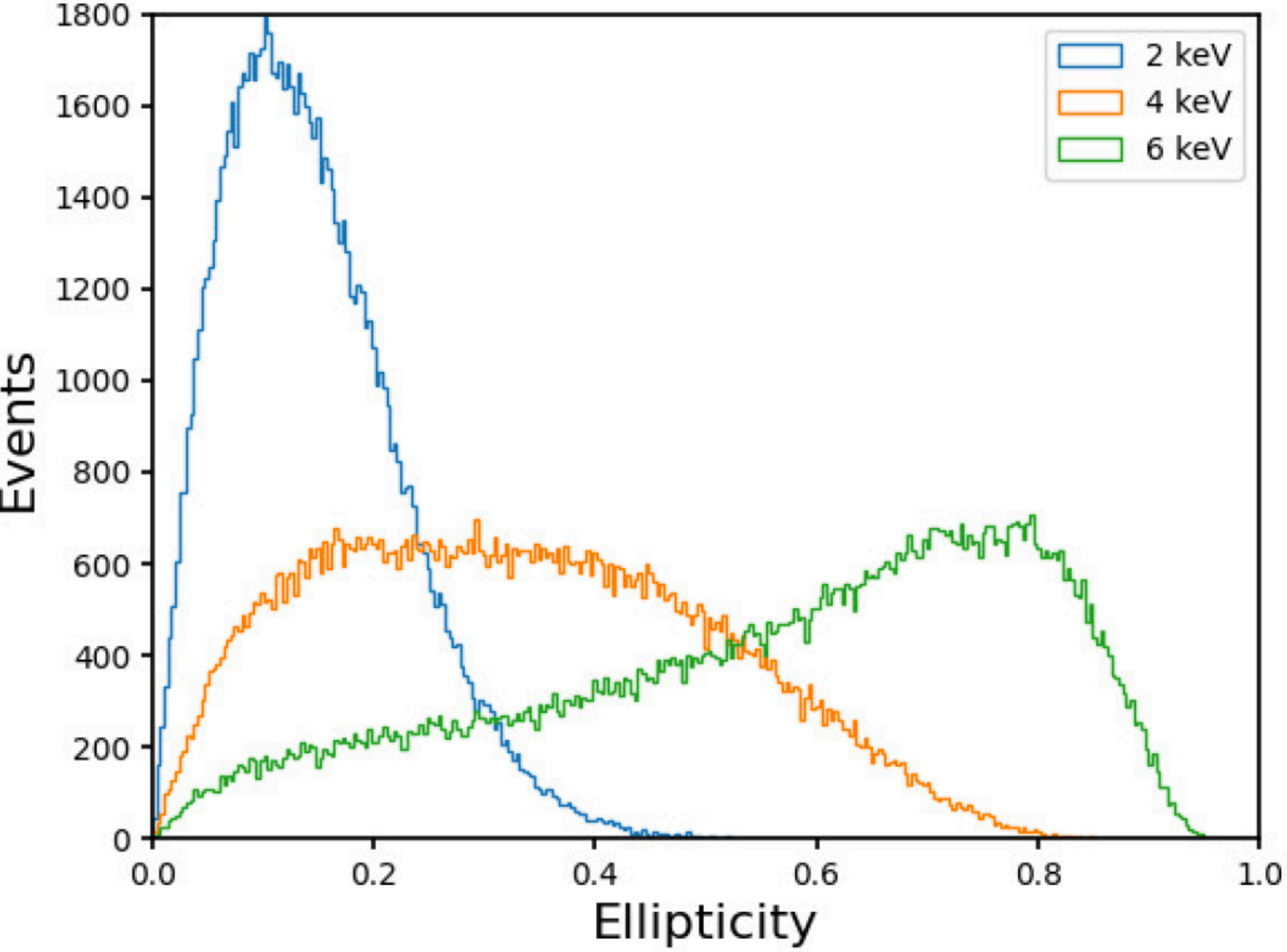
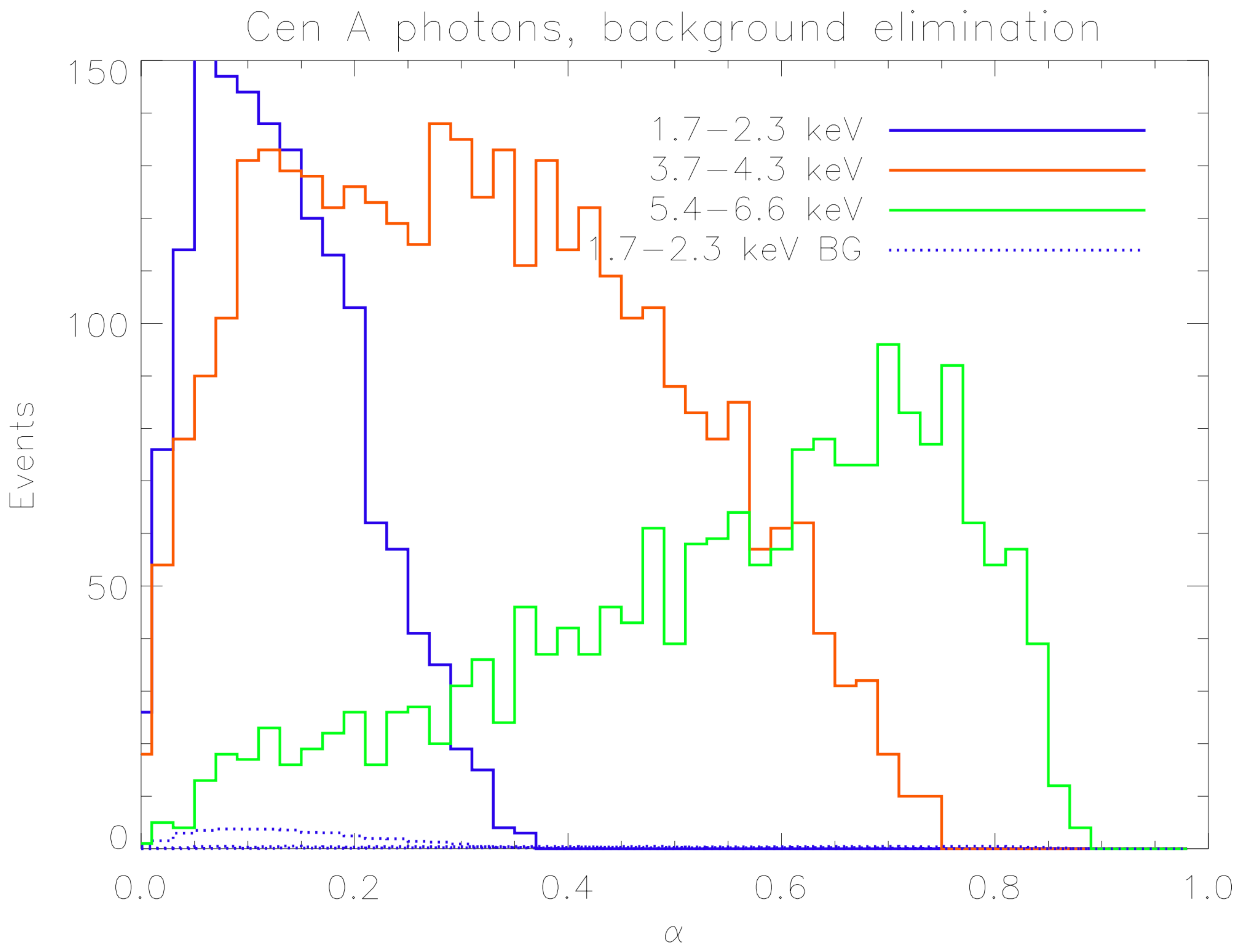




Comparing α distributions

- Left: Fig. 4 of Di Marco+ (2022), simulated data, exact energies (2, 4, 6 keV)
- Right: Cen A, inferred energies in 'narrow' bands about 2, 4, 6 keV





Updating XSPEC analysis

- New model is $Q_j(E,\Theta) = T\int A(E')\mathcal{Q}(E',\Theta)\mathcal{M}_j(E',E)dE', \quad U_j(E,\Theta) = T\int A(E')\mathcal{U}(E',\Theta)\mathcal{M}_j(E',E)dE'$
  - Index j refers to specific values of α_j
  - New detector mRMF is $\mathcal{M}_j(E',E) = \mu(\alpha_j,E')\epsilon(E')\phi(\alpha_j,E')R(E',E)$ New!
  - where $\sum_j \phi(\alpha_j,E') = 1$ and $\sum_j \mu(\alpha_j,E')\phi(\alpha_j,E') = \mu(E')$ (unweighted, uncut)
- Original: $\lambda(n_0,\Pi,\varphi;E,\psi) = [1+\Pi\mu_E\cos(2\psi+2\varphi)]n(E')A(E')TdE'd\psi$
  - gives MDP99 $= 4.29/\sqrt{\sum \mu_{E_i}^2 C(E_i)}$
- Then $\lambda(n_0,\Pi,\varphi;E,\alpha_j,\psi) = \int dE'[1+\Pi\mathcal{M}_j(E',E)\cos(2\psi+2\varphi)]n(E')A(E')Td\psi$ and

$$\tilde{S}(q,u) = -2\sum_i \ln(1+q\mu(\alpha_i,E_i)\cos 2\psi_i + u\mu(\alpha_i,E_i)\sin 2\psi_i)$$

Likelihood Formulation (Marshall 2021)

See 12/1/20 CalStats WG presentation

- Expected counts in $dEd\psi$ in time *T* for $Q = qI, U = uI$:
  - $\lambda(E, \psi; n_0, q, u)dEd\psi = [1 + \mu_E(q\cos 2\psi + u\sin 2\psi)]n_E A_E T dEd\psi$, where
    - $A_E = A\alpha(E)$ is the instrument effective area (independent of q or u by definition)
    - $n_E = n_0\phi(E)$ has units of ph/cm2/s/keV per unit (measured) phase angle, ψ
  - Require $\Pi^2 \equiv q^2 + u^2 \leq 1$ physically (Π is fractional linear polarization)
  - Define $\phi_0 = \tan^{-1}(u/q) = 2\varphi$
- Likelihood: $S(n_0, q, u) = -2\ln\mathscr{L} = -2\sum_i \ln\lambda(E_i, \psi_i) + 2T\int f_E A_E dE \int_0^{2\pi} [1 + \mu(E)(q\cos 2\psi + u\sin 2\psi)]d\psi$ or $\tilde{S}(q, u) = -2\sum_i \ln(1 + q\mu_i\cos 2\psi_i + u\mu_i\sin 2\psi_i)$
- MDP99 $= 4.29/\sqrt{\sum \mu_E^2 C(E)}$ for small $\Pi\mu$

Adding BG to Likelihood

- $\lambda_S(\psi) = \frac{1}{2\pi}\{n_0[1 + \mu(q\cos 2\psi + u\sin 2\psi)] + \zeta B\}$ for the $N = C_S + C_B$ counts in the source region, $\lambda_B(\psi) = \frac{B}{2\pi}$ for the N_B counts in the BG region, and $C_B = \zeta B$ is the expected BG in the source region
- Likelihood: $S(n_0, q, u) = -2\sum_{i=1}^{N} \ln[n_0(1 + qc_i + us_i) + \zeta B] + 2n_0 + 2B(1+\zeta) - 2N_B \ln B$, minimized to give $\hat{B} = N_B$ and 3 equations to solve mutually (and numerically): $\hat{n}_0 = \sum w_i,\ \ 0 = \sum w_i c_i,$ and $\ 0 = \sum w_i s_i$, where $w_i = [1 + \hat{q}c_i + \hat{u}s_i + \zeta N_B/n_0]^{-1}$
- $\mathrm{MDP}_{99} = \frac{4.29\sqrt{C_S + C_B}}{C_S\sqrt{<\mu_i^2>}}$ for unpolarized data, similar to Elsner+ (2012) result.

Azimuthally Nonuniform Response

- Define $w(\psi)$ as fraction of exposure to angle ψ
  - e.g. Bragg reflector, where $w(\psi) = \delta(\psi - \psi_0)$ or $w(\psi) = \sum_i \delta(\psi - \psi_i)$
- Count density: $\lambda(\psi) = [1 + \mu_E w(\psi)(q \cos 2\psi + u \sin 2\psi)] n_E A_E T d\psi$
- Likelihood: $S(n_0, q, u) = -2N \ln n_0 - 2\sum_i \ln(1 + q\mu_i\alpha_i + u\mu_i\beta_i) + 2Kn_0 + 2K_\mu n_0 A q + 2K_\mu n_0 B u$, where

  $\alpha(\psi) \equiv w(\psi)\cos 2\psi$, $\beta(\psi) \equiv w(\psi)\sin 2\psi$, $A \equiv \int \alpha(\psi) d\psi$, and $B \equiv \int \beta(\psi) d\psi$, and

  $K = \frac{2\pi T}{n_0} \int n(E;\xi) A_E dE$ and $K_\mu = \frac{T}{n_0} \int n(E;\xi) A_E \mu_E dE$ are treated as uninteresting constants
- If $w(\psi) \neq f(\psi)$ or $\psi_i = \psi_0 + \frac{2\pi i}{n}$ (so $A = B = 0$) or if $q = u = 0$, then $n_0 = N$
- All parameters are covariant otherwise

Summary

- Polarimetry adds complexity to spectral fits
- xspec: updated for simple case
- Real IXPE data require mRMFs in E and ellipticity α
- Accounting for background is a simple addition
- Important to design instruments with uniform $w(\psi)$